Due Sun

1.8 – Introduction to Linear Transformations

$\boldsymbol{R}^2$ is the set of all ordered pairs, $\boldsymbol{R}^3$ is the set of all ordered triples, and $\boldsymbol{R}^n$ is the set of all ordered n-tuples. Elements of R^n are called **vectors**.

$\mathbb{R}^n$

The **standard basis vectors** for R^n are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

These can also be written as row vectors if convenient.

All vectors in $\mathbb{R}^n$ can be expressed as linear combinations of these.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots x_n\vec{e}_n$$

A **function** f is a rule that associates with an input (an element of the **domain** of f) a unique output (an element of the **codomain** of f). The output is the **image** of the input, and the set of all images is called the **range** of f. Note that the range of f is a subset of the codomain of f.

$f(a)=b$ is the image of a under f.

$a \in$ domain of f $\quad$ $b \in$ codomain of f

Consider $f(x) = |x|$

Domain: $\mathbb{R}$
Codomain: $\mathbb{R}$
Range: $[0, \infty)$

The range is a subset of the codomain.

Transformations

Consider the system

$$4 = 2x - 3y + 5z$$
$$6 = 7x + 2y - z$$

In matrix form, this is

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The 2×3 matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix}$ maps vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

A is a transformation matrix from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Definition 1: If T is a function with domain R^n and codomain R^m, then we say that T is a **transformation** from R^n to R^m or that T **maps** from R^n to R^m, which we denote by writing $T: R^n \to R^m$. In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n.

5. Find the domain and codomain of the transformation defined by the matrix product.

a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Dom: R^3
Codom: R^2

Dom: R^2
Codom: R^3

A **matrix transformation** $\mathbf{w} = A\mathbf{x}$ maps a vector $\mathbf{x} \in R^n$ to a vector $\mathbf{w} \in R^m$ by multiplying $\mathbf{x}$ on the left by A [which is an $m \times n$ matrix]. If $m = n$, then we call the transformation a **matrix operator**. A matrix transformation is denoted $T_A: R^n \to R^m$ or $\mathbf{w} = T_A(\mathbf{x})$ if we do not need to specify the domain and codomain. This can also be written in the form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

verbalized as "T_A maps $\mathbf{x}$ into $\mathbf{w}$." The matrix A is the **standard matrix** for the transformation.

8. Find the domain and codomain of the transformation T defined by the formula.

a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$

b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

a. Dom: R^4
Codom: R^2

b. Dom: R^3
Codom: R^3 $\Big\}$ operator

12. Find the standard matrix for the transformation defined by the equations.

a.

$$\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$$

$$\begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$$

b.

$$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

13. Find the standard matrix for the transformation T defined by the formula.

a. $T: R^2 \Rightarrow R^2$

a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

b. $T(x_1, x_2, x_3) = (4x_1 + x_2, 5x_1 + x_2)$

a. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[T] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

b. $[T] = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix}$

The **zero transformation** from R^n to R^m, $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$, maps every vector in R^n to the zero vector in R^m.

The **identity operator** $T_{I_n}(\mathbf{x}) = I_n(\mathbf{x}) = \mathbf{x}$ maps every vector in R^n to itself.

Theorem 1.8.1 Properties of Matrix Transformations

For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

a) $T_A(\mathbf{0}) = \mathbf{0}$

b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

These follow from matrix arithmetic

Theorem 1.8.2 $T: R^n \to R^m$ is a matrix transformation $\boxed{\text{if and only if}}$ $\Longleftrightarrow$ the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

i) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

<u>Pf</u>: ($\Rightarrow$) By Thm 1.8.1 (b) & (c).

($\Leftarrow$) Assume i & ii hold. The proof will be complete if there is an $m \times n$ matrix A such that $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

By (i) & (ii), $T(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_n\vec{u}_n)$

$$= k_1T(\vec{u}_1) + k_2T(\vec{u}_2) + \cdots + k_nT(\vec{u}_n)$$

for all vectors $\vec{u}_i \in R^n$ and all scalars k_i.

In particular, this holds for $\vec{u}_i = \vec{e}_i$, which gives $k_1T(\vec{e}_1) + k_2T(\vec{e}_2) + \cdots k_nT(\vec{e}_n)$, where $\vec{e}_i$ is a standard basis vector for $\mathbb{R}^n$. This is

$$A\vec{x} = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)\right] \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

So $A\vec{x} = T(k_1\vec{e}_1 + k_2\vec{e}_2 \cdots k_n\vec{e}_n) = T(\vec{x})$

by linearity of T. ✓

Ex: $A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 5 & 8 \end{bmatrix}$ $\quad A\vec{e}_1 = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$A\vec{e}_2 = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 5 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

22. Use Theorem 1.8.2 to show that T is a matrix transformation.

a. $T(x, y, z) = (x + y, y + z, x)$

b. $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

we need to show that i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

ii) $T(k\vec{u}) = kT(\vec{u})$

a. Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

Then $\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$

$T(\vec{u}+\vec{v}) = (u_1+v_1+u_2+v_2,\ u_2+v_2+u_3+v_3,\ u_1+v_1)$

$= (u_1+u_2,\ u_2+u_3,\ u_1) + (v_1+v_2,\ v_2+v_3,\ v_1)$

$= T(\vec{u}) + T(\vec{v})$ ✓

ii) Let $\vec{u}$ be as in part (a) and k a scalar.

Then $k\vec{u} = (ku_1, ku_2, ku_3)$

and $T(k\vec{u}) = (ku_1 + ku_2,\ ku_2 + ku_3,\ ku_1)$

$= k(u_1 + u_2,\ u_2 + u_3,\ u_1)$

$= kT(\vec{u})$

So T is a matrix transformation.

23. Use Theorem 1.8.2 to show that T is not a matrix transformation.

a. $T(x, y) = (x^2, y)$

b. $T(x, y, z) = (x, y, xz)$

a. Let $k \in \mathbb{R}$.

Then $T(k(x,y)) = T(kx, ky) = (k^2x^2, ky)$

$= k(kx^2, y)$

But $kT(x,y) = k(x^2, y) \neq T(k(x,y))$

We saw in $(\Leftarrow)$ of proof of Thm 1.8.2 that the standard matrix for a transformation denoted $[T_A] = A = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)]$.

A **linear transformation** $T: R^n \rightarrow R^m$ possesses the two **linearity conditions**, those being:

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k,

i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)

Theorem 1.8.3 Every linear transformation from R^n to R^m is a matrix transformation, and conversely every matrix transformation from R^n to R^m is a linear transformation.

can be used interchangably

28. The images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \rightarrow R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix} = T_A(\vec{x})$$

Aside: Note

$$A\vec{e}_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Given $\vec{u} = a\vec{e}_1 + b\vec{e}_2$, (lin. comb. of std. basis vectors)

$$T(\vec{u}) = aT(\vec{e}_1) + bT(\vec{e}_2)$$

Likewise, if $\vec{e}_1 = a\vec{u} + b\vec{v}$, then $T(\vec{e}_1) = aT(\vec{u}) + bT(\vec{v})$

Ex: Find the standard matrix A for the linear transformation $T: R^2 \to R^3$ for which $T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$ and use it to compute $T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$.

$\vec{u}$ $\qquad$ $\vec{v}$

Such that

Want $a, b \ \ni \ \vec{e}_1 = a\vec{u} + b\vec{v}$

and another $a, b \ \ni \ \vec{e}_2 = a\vec{u} + b\vec{v}$

$a\vec{u} + b\vec{v} = \vec{e}_1 : \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} a + \begin{bmatrix} 3 \\ -5 \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \left(= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

a, b for $\vec{e}_1$

a, b for $\vec{e}_2$

$$\begin{array}{cccc} \vec{u} & \vec{v} & \vec{e}_1 & \vec{e}_2 \end{array}$$
$$\left[\begin{array}{cc|c|c} -1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$\vec{e}_1 = 5\vec{u} + 2\vec{v}, \qquad \vec{e}_2 = 3\vec{u} + 1\vec{v}$

$T(\vec{e}_1) = 5T(\vec{u}) + 2T(\vec{v})$

$T(\vec{e}_1) = 5\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 20 \\ -9 \\ 2 \end{bmatrix}$

$T(\vec{e}_2) = 3T(\vec{u}) + T(\vec{v})$

$T(\vec{e}_2) = 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix}$

$T_A: R^2 \to R^3$

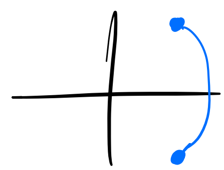
$$A = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}$$

3×2

$$T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -47 \\ 24 \\ -5 \end{bmatrix}$$

Theorem 1.8.4 If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transformations, and if
$T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

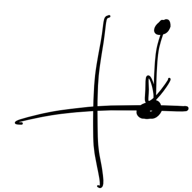
Matrix operators on R^2



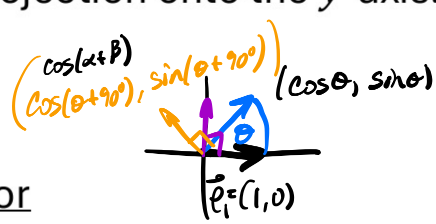
Reflection operators

- Reflection about the x-axis: $T(x, y) = T(x, -y)$, $T_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection about the y-axis: $T(x, y) = T(-x, y)$, $T_A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection about the line $y = x$: $T(x, y) = (y, x)$, $T_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Projection operators

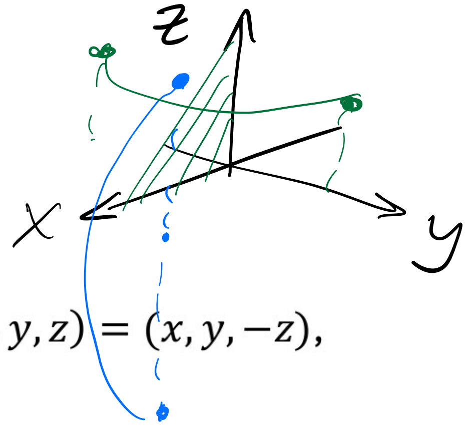


- Orthogonal projection onto the x-axis: $T(x, y) = T(x, 0)$,
  $T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Orthogonal projection onto the y-axis: $T(x, y) = T(0, y)$,
  $T_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



Rotation operator

- Counterclockwise rotation about the origin through an angle θ:
  $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Matrix operators on R^3

Reflection operators

- Reflection about the xy-plane: $T(x, y, z) = (x, y, -z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Reflection about the xz-plane: $T(x, y, z) = (x, -y, z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Reflection about the yz-plane: $T(x, y, z) = (-x, y, z)$,

$$T_A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection operators

- Orthogonal projection onto the xy-plane: $T(x, y, z) = (x, y, 0)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Orthogonal projection onto the xz-plane: $T(x, y, z) = (x, 0, z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Orthogonal projection onto the yz-plane: $T(x, y, z) = (0, y, z)$,

$$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$